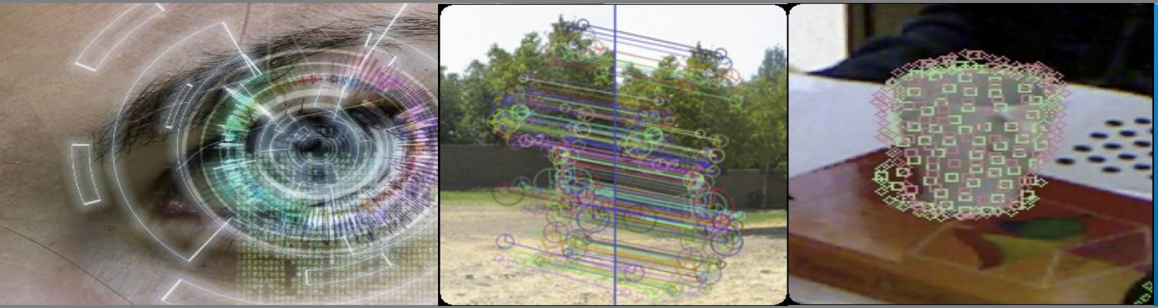
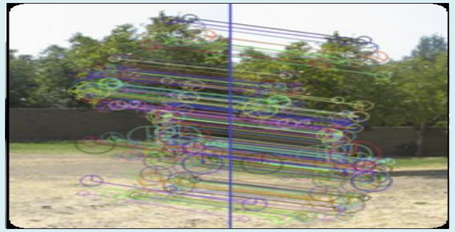
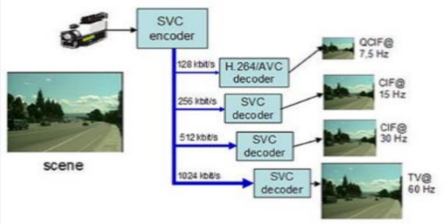
Adaptive Signal Processing and System Theory
(#2: The z-transform)



2023 Spring

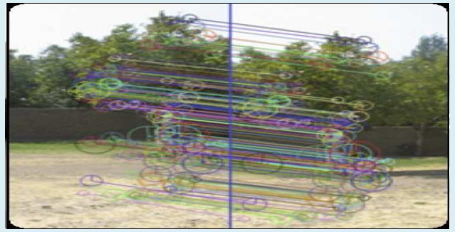
Prof. Byung-Gyu Kim
Intelligent Vision Processing Lab. (IVPL)
http://ivpl.sookmyung.ac.kr
Dept. of IT Engineering, Sookmyung Women's University
E-mail: bg.kim@sookmyung.ac.kr

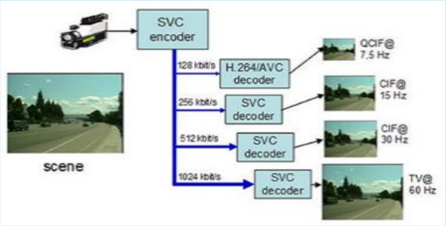




Contents

- Introduction of the z-Transform
- Relationship of Properties of Seq. to Its z-Transform
- Z-Transform for LTI system





Contents

- **Introduction of the z-Transform**
- Relationship of Properties of Seq. to Its z-Transform
- Z-Transform for LTI system

Introduction

- z-Transform representation of a sequence
  - Fourier transform does not converge for all sequences.
    → Generalization of Fourier Transform (FT)
  - In analytical problems, the z-transform notation is often more convenient than FT notation.

z-Transform (1)

- Fourier transform (FT)

$$X(e^{j\omega}) = \sum x[n]e^{-j\omega n}$$

- z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$: **bilateral z-transform**

where z = a constant complex variable.

$$x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(Z)$$

- **Unilateral z-transform**

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

z-Transform (2)

- In general, $z = re^{j\omega}$ (since z is complex value,)

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

- **Fourier Transform of product of the original $x[n]$ and r^{-n} .**
- If r=1, then FT of $x[n]$.

For unit circle in z-plane, we can evaluate $X[z]$ at points on the unit circle:

→ Fourier Transform for $0 \leq \omega \leq \pi$

z-Transform (3)

❖ Region of Convergence (ROC)

- FT and z-transform do not converge for all sequences.
- For any seq., the set of values z for which the z-transform power converge is called "***the region of convergence*** (ROC)".

If absolutely summable, FT converges to ~~~~.

$$|X(z)| < \infty \quad \text{if} \quad \sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$$

From the above, the convergence depends on $|z|$ ***!!!***

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

: Power series or Laurent series

❖ Region of Convergence (ROC)

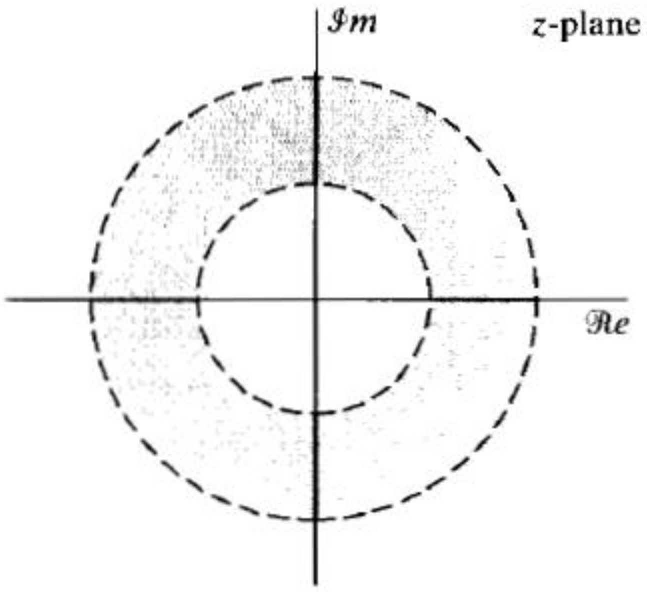


Figure 3.2 The region of convergence (ROC) as a ring in the *z*-plane. For specific cases, the inner boundary can extend inward to the origin, and the ROC becomes a disc. For other cases, the outer boundary can extend outward to infinity.

z-Transform (5)

❖ Poles and Zeros

- In case of z-transform,

$$X(z) = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials in z .

- **Poles**: z value of $X(z) = 0$.
- **Zeros**: z value of $X(z) = \infty$.

Important relationships exists between the locations of poles of $X(z)$ and The ROC of the z-transform.

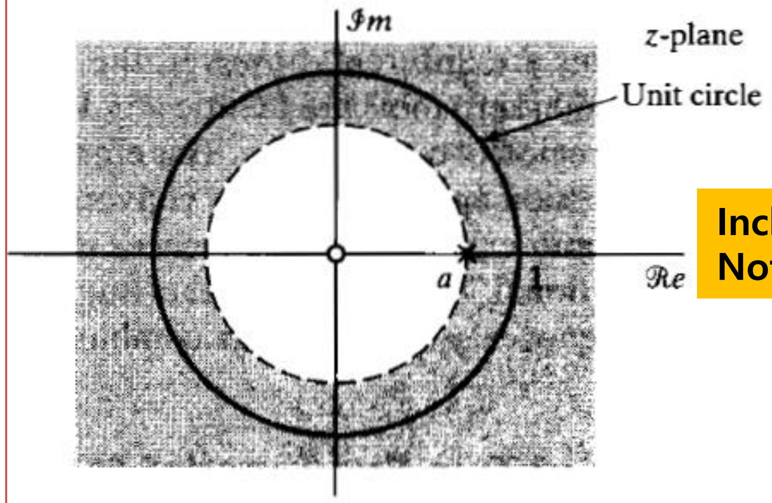
- Ex 3.1) signal $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence,

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n < \infty$$

Then, $X(z) = \dfrac{1}{1 - az^{-1}}$ for $|az^{-1}| < 1(|z| > |a|).$



Include unit-circle : sequence converges
Not include unit-circle: seq. does not converge

Figure 3.3 Pole–zero plot and region of convergence for Example 3.1.

z-Transform (7)

- Ex 3.3) sum of two exponential sequences

$$x[n] = (\frac{1}{2})^n u[n] + (-\frac{1}{3})^n u[n]$$

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}$$

For convergence of X(z), $|\frac{1}{2}z^{-1}| < 1 \text{ and } |-\frac{1}{3}z^{-1}| < 1$

$$\therefore \quad |z| > \frac{1}{2}$$

(ROC can be plotted as the next…!!!)

z-Transform (8)

- ROC of the previous slide

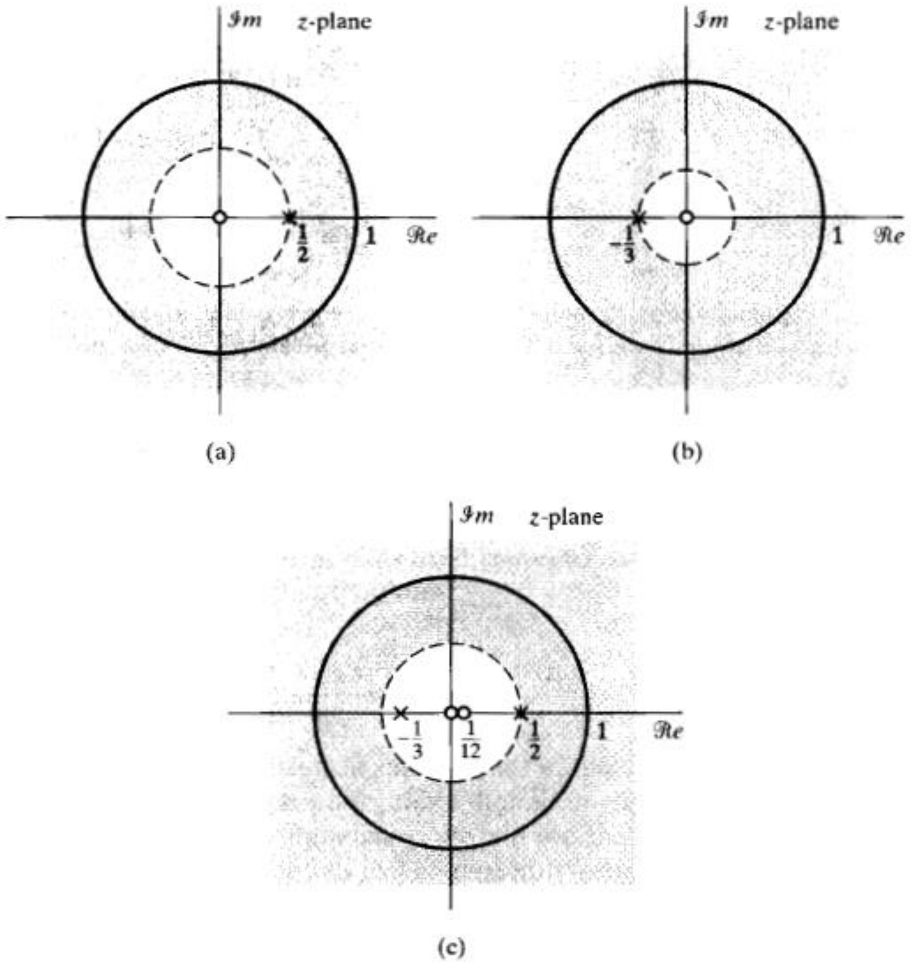


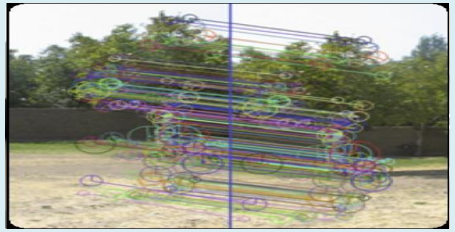
Figure 3.5 Pole–zero plot and region of convergence for the individual terms and the sum of terms in Examples 3.3 and 3.4. (a) $1/(1-\frac{1}{2}z^{-1})$, $|z| > \frac{1}{2}$. (b) $1/(1+\frac{1}{3}z^{-1})$, $|z| > \frac{1}{3}$. (c) $1/(1-\frac{1}{2}z^{-1}) + 1/(1+\frac{1}{3}z^{-1})$, $|z| > \frac{1}{2}$.

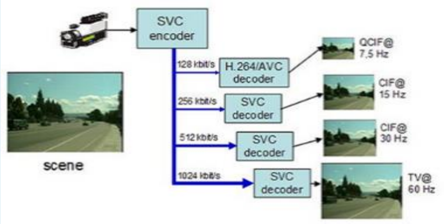
z-Transform (9)

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

IVPL Intelligent Vision Processing Lab





Contents

- Introduction of the z-Transform
- Relationship of Properties of Seq. to Its z-Transform
- Z-Transform for LTI system

Properties of the ROC for z-transform (1)

❖ Properties of the ROC

- ***Usually, the properties of the ROC depend on the nature of the signal….!!!***
- ***Property 1 ~ 8***

The Inverse z-Transform (1)

- Inverse z-transform
  - Is represented as complex contour integral:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz,$$

where C represents a closed contour within the ROC of the z-transform.

- Methods for Inverse z-transform
  - Inspection Method
    - By inspection certain transform pairs.
    - For example, $a^n u[n] \longleftrightarrow \dfrac{1}{1 - az^{-1}}$, where $|z| > |a|$.

    Then the inverse of $X(z) = \dfrac{1}{1 - \frac{1}{2} z^{-1}}$ equals ?

- Partial Fraction Expansion
  - Sometimes, $X(z)$ may not be given explicitly in an available table. Let us assume that $X(z)$ = a ratio of polynomials like

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{z^N \sum_{k=0}^{M} b_k z^{-k}}{z^M \sum_{k=0}^{N} a_k z^{-k}}$$

    - If M>N, then (M-N) poles at z=0.
    - If N>M, then (N-M) zeros at z=0.
  - Also,

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

where c_k = nonzero zeros of $X(z)$, d_k = nonzero poles of $X(z)$.

The Inverse z-Transform(3)

- Case 1) If M<N and poles are all the 1-st order, then

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

By multiplying $(1 - d_k z^{-1})$ and $z = d_k$,

$$\therefore \quad A_k = (1 - d_k z^{-1}) X(z)|_{z=d_k}$$

- Ex 3.9) 2-nd order z-transform

$$X(z) = \frac{1}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{2} z^{-1})} \quad |z| > \frac{1}{2}$$

Let

$$X(z) = \frac{A_1}{(1 - \frac{1}{4} z^{-1})} + \frac{A_2}{(1 - \frac{1}{2} z^{-1})} \quad |z| > \frac{1}{2}$$

The Inverse z-Transform(4)

After computing A_1 and A_2,

$$X(z) = \frac{-1}{(1-\frac{1}{4}z^{-1})} + \frac{2}{(1-\frac{1}{2}z^{-1})} \quad |z| > \frac{1}{2}$$

From Table 3.1 and the linearity of z-transform,

$$\therefore \ x[n] = 2(\frac{1}{2})^n u[n] - (\frac{1}{4})^n u[n].$$

- In case of M $\geq$ N, the complete partial fraction expression would be

$$X(z) = \underbrace{\sum_{r=0}^{M-N} B_r z^{-r}}_{\text{Normal parts}} + \underbrace{\sum_{k=0}^{N} \frac{A_k}{1-d_k z^{-1}}}_{\text{Fraction parts}}$$

The Inverse z-Transform(5)

- Ex 3.10) Inverse by partial fraction

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} \quad |z| > \frac{1}{2}$$

Let $X(z) = B_0 + \frac{A_1}{(1-\frac{1}{2}z^{-1})} + \frac{A_2}{(1-2z^{-1})} \quad |z| > \frac{1}{2}$

$$\therefore \ B_0 = 2$$ (by direct division)

Then,

$$X(z) = 2 + \frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$$

Now for coefficient A_1 and A_2, by $A_k = (1-d_k z^{-1})X(z)|_{z=d_k}$

$$A_1 = (2 + \frac{-1+5z^{-1}}{(1-\frac{1}{2}z^{-1})(1-z^{-1})})(1-\frac{1}{2}z^{-1})|z = \frac{1}{2} = -9$$

$$A_2 = 8$$

Therefore,

The Inverse z-Transform(6)

From Table 3.1,

$$\therefore \; X(z) = 2 - \frac{9}{(1-\frac{1}{2}z^{-1})} + \frac{8}{(1-z^{-1})}$$

$$\begin{aligned} 2 &\longleftrightarrow 2\delta[n], \\ \frac{1}{1-\frac{1}{2}z^{-1}} &\longleftrightarrow (\frac{1}{2})^n u[n], \\ \frac{1}{1-z^{-1}} &\longleftrightarrow u[n], \end{aligned}$$

From the linearity of the z-transform,

$$\therefore \; x[n] = 2\delta[n] - 9(\frac{1}{2})^n u[n] + 8u[n].$$

- Power Series Expansion

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + \cdots$$

Ex 3.11) Finite-length Series

Suppose $X(z) = z^2(1 - \frac{1}{2}z^{-1})(1 + z^{-1})(1 - z^{-1}),$

Power series evaluation and order matching

$$\therefore\ x[n] = \delta[n] - \frac{1}{2}\delta[n] - \delta[n] + \frac{1}{2}\delta[n-1].$$

It is very useful when $X(z)$ is the ratio of polynomials…!!!!!

IVPL Intelligent Vision Processing Lab

z-transform Properties (1)

❖ For LTI system,

$$x_1[n] \longleftrightarrow X_1(z), \quad ROC = R_{x_1}$$

$$x_2[n] \longleftrightarrow X_2(z), \quad ROC = R_{x_2}$$

- Linearity

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(z) + bX_2(z), \ \text{ROC contains } R_{x_1} \cap R_{x_2}.$$

- Time shifting

$$x[n-n_0] \longleftrightarrow Z^{-n_0}X(z)$$

- Multiplication by an Exponential Sequence

$$z_0^n x[n] \longleftrightarrow X(z/z_0), \ \text{ROC} = |z_0|R_x.$$

- Differentiation of $X(z)$

$$nx[n] \longleftrightarrow -z\frac{X(z)}{dz}, \ \text{ROC} = R_x.$$

How to prove????

z-transform Properties (2)

- Conjugation of a Complex Sequence

$$x^*[n] \longleftrightarrow X^*(z^*), \quad ROC = R_x$$

- Time-Reversal

$$x^*[-n] \longleftrightarrow X^*(1/z^*), \quad ROC = \frac{1}{R_x}$$

If $x[n]$ is real, then

$$x[-n] \longleftrightarrow X(1/z), \quad ROC = \frac{1}{R_x}$$

- Convolution of Sequence

$$x_1[n] * x_2[n] \longleftrightarrow X_1(z)X_2(z), \text{ ROC contains } R_{x_1} \cap R_{x_2}.$$

- Some z-transform properties

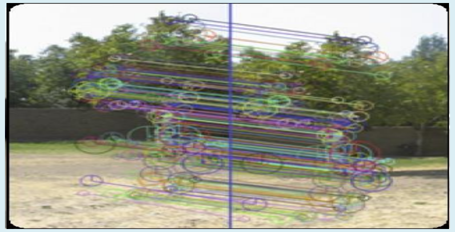
(Refer to Table 3.2....!!!)

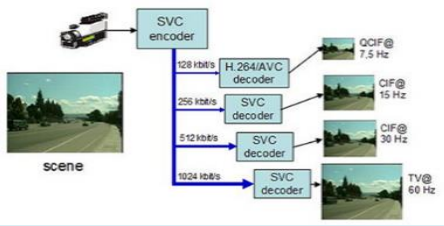
z-transform Properties (3)

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Section Reference	Sequence	Transform	ROC
	$x[n]$	$X(z)$	R_x
	$x_1[n]$	$X_1(z)$	R_{x_1}
	$x_2[n]$	$X_2(z)$	R_{x_2}
3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
3.4.2	$x[n - n_0]$	$z^{-n_0} X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\| R_x$
3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
	$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
	$\mathcal{J}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
3.4.8	Initial-value theorem:		
	$x[n] = 0, \quad n < 0$ $\quad \lim_{z\to\infty} X(z) = x[0]$		

Intelligent Vision Processing Lab





Contents

- Introduction of the z-Transform
- Relationship of Properties of Seq. to Its z-Transform
- **Z-Transform for LTI system**

z-Transform and LTI systems (1)

❖ LTI systems

$$x[n] \longrightarrow \boxed{\text{system } h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$

$$Y(z) = H(z)X(z) \quad \therefore \; H(z) = \frac{Y(z)}{X(z)}$$: ***system function (transfer function) of LTI system***

Ex 3.20) Convolution of ~

$$x[n] = Au[n], \quad h[n] = a^n u[n]$$

$$H(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}}, \quad |z| > |a|. \qquad X(z) = \frac{A}{1 - z^{-1}}, \quad |z| > 1.$$

$$Y(z) = H(z)X(z) =$$

z-Transform and LTI systems (2)

$$= \frac{A}{(1-az^{-1})(1-z^{-1})},$$

$$= \frac{A}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right), \quad |z| > 1$$

Inverse of LTI system

$$\therefore \quad y[n] = \frac{A}{1-a}(1-a^{n+1})u[n].$$

❖ Some cases, LTI system function:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{N} a_k y[n-k]$$

The Unilateral z-Transform

❖ Unilateral or One-sided z-Transform

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

(Plz, read the details in pp. 164-166)

HW#2

- In textbook, Chap. 3,
  - Prob. 3.4, 3.7, 3.9, 3.17, 3.29

- Due day: ~ to the next week.

Thank you for your attention.!!!
QnA

http://ivpl.sookmyung.ac.kr